Chapter 6

Lecture 3

Legendre Transformations Lagrange & Poisson Bracket

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6.3 Legendre Transformations

Lagrangian of the system $L = L(q_{i,} \dot{q}_i, t)$

Lagrange's equation $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$

Where momentum $p_i = \frac{\partial L(q_{i,} \dot{q}_i, t)}{\partial \dot{q}_i}$

In Lagrangian to Hamiltonian transition variables changes from $(q, \dot{q}_i, t)$ to (q, p, t), where p is related to q and $\dot{q}_i$ by above equation.

This transformation process is known as Legendre transformation.

Consider a function of only two variable $f(x, y)$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$$df = udx + vdy$$

6.3 Legendre Transformations

If we wish to change the basis of description from x, y to a new set of variable y,u

Let $g(y, u)$ be a function of u and y defined as $g = f - ux$ (I)

Therefore $$dg = df - xdu - udx$$

since $$df = udx + vdy$$

So $$dg = udx + vdy - xdu - udx$$

$$dg = vdy - xdu$$

And the quantities v and x are function of y and u respectively.

And $$dg = \frac{\partial g}{\partial y}dy + \frac{\partial g}{\partial u}du$$

$$\frac{\partial g}{\partial y} = v \qquad \& \qquad \frac{\partial g}{\partial u} = -x$$

6.3 Legendre Transformations

Equation I represent Legendre transformation. The Legendre transformation is frequently used in thermodynamics as

$$dU = TdS - PdV \qquad \text{for } U(S,V)$$

And $$H = U + PV \qquad H(S,P)$$

$$dH = TdS + VdP$$

And $$F = U - TS \qquad F(T,V)$$

$$G = H - TS \qquad G(T,P)$$

U = Internal energy T = Temperature S = Entropy P = Pressure

V = Volume H = Enthalpy F = Helmholtz Free Energy G=Gibbs free energy

6.3 Legendre Transformations

The transformation of $(q, \dot{q}, t)$ to (q, p, t) is however different.

Since $$dL = \frac{\partial L}{\partial q} dq + \frac{\partial L}{\partial \dot{q}} d\dot{q} + \frac{\partial L}{\partial t} dt$$

And $$\frac{\partial L}{\partial \dot{q}} = p \qquad \Rightarrow \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \dot{p}$$

Therefore, $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \frac{d}{dt} p - \frac{\partial L}{\partial q} = 0$$

$$\Rightarrow \frac{\partial L}{\partial q} = \dot{p}$$

Therefore , $$dL = \frac{\partial L}{\partial q} dq + \frac{\partial L}{\partial \dot{q}} d\dot{q} + \frac{\partial L}{\partial t} dt$$

$$dL = \dot{p} dq + p d\dot{q} + \frac{\partial L}{\partial t} dt$$

And $$H = \dot{q} p - L$$

6.4 Lagrange's Bracket

Lagrange brackets were introduced by Joseph Louis Lagrange in 1808–1810.

- Mathematical formulation of Classical Mechanics.
- The Lagrange bracket is not use in modern mechanics
- The Lagrange's bracket are defined as if u and v are functions depending on q_i Zand p_i then

$$\{u, v\}_{q,p} = \sum_i \left(\frac{\partial q_i}{\partial u} \frac{\partial p_i}{\partial v} - \frac{\partial q_i}{\partial v} \frac{\partial p_i}{\partial u} \right) = \sum_i \frac{\partial(q_i, p_i)}{\partial(u, v)}$$

And invariant under canonical transformation.

6.4 Lagrange's Bracket

Some properties

i. $\{u, v\}_{q,p} = -\{v, u\}_{q,p}$ — Lagrange's bracket are anti commutative

ii. $\{q_i, q_j\}_{q,p} = 0 = \{p_i, p_j\}_{q,p}$ — For identical function it is zero

iii. $\{q_i, p_j\}_{p,q} = \delta_{ij} \quad \begin{cases} 1 \; if \; i = j \\ 0 \; if \; i \neq j \end{cases}$ — Equal to Kronecker delta function.

iv. $\{u, v\}_{q,p} = \{u, v\}_{Q,P}$ — Invariance of Lagrange Bracket

6.4 Lagrange's Bracket

Proof: $\{u, v\}_{q,p} = -\{v, u\}_{q,p}$

$$\{u, v\}_{q,p} = \sum_i \left(\frac{\partial q_i}{\partial u}\frac{\partial p_i}{\partial v} - \frac{\partial q_i}{\partial v}\frac{\partial p_i}{\partial u}\right)$$

Taking negative common $\{u, v\}_{q,p} = -\sum_i \left(-\frac{\partial q_i}{\partial u}\frac{\partial p_i}{\partial v} + \frac{\partial q_i}{\partial v}\frac{\partial p_i}{\partial u}\right)$

$$\{u, v\}_{q,p} = -\sum_i \left(\frac{\partial q_i}{\partial v}\frac{\partial p_i}{\partial u} - \frac{\partial q_i}{\partial u}\frac{\partial p_i}{\partial v}\right)$$

$\{u, v\}_{q,p} = -\{v, u\}_{q,p}$ ***proved***

6.4 Lagrange's Bracket

Proof: $\{q_i, q_j\}_{p,q} = 0 = \{p_i, p_j\}_{q,p}$ For identical function it is zero

$$\{q_i, q_j\}_{q,p} = \sum_k \left(\frac{\partial q_k}{\partial q_i}\frac{\partial p_k}{\partial q_j} - \frac{\partial q_k}{\partial q_i}\frac{\partial p_k}{\partial q_j}\right)$$

Here p and q are treated as independent co-ordinates in phase space. So

$$\frac{\partial p_k}{\partial q_j} = 0 = \frac{\partial p_k}{\partial q_i}$$

Therefore $\{q_i, q_j\}_{q,p} = 0$

similarly , we can show $\{p_i, p_j\} = 0$

6.4 Lagrange's Bracket

$\{q_i, p_j\} = \delta_{ij}$ $\begin{cases} 1 \; if \; i = j \\ 0 \; if \; i \neq j \end{cases}$ Equal to Kronecker delta function.

$$\{q_i, p_j\}_{q,p} = \sum_k \left(\frac{\partial q_k}{\partial q_i} \frac{\partial p_k}{\partial p_j} - \frac{\partial q_k}{\partial p_j} \frac{\partial p_k}{\partial q_i} \right)$$

0

Since p, q are independent, we get $\frac{\partial q_k}{\partial p_j} = \frac{\partial p_k}{\partial q_i} = 0$

Therefore $\{q_i, p_j\}_{q,p} = \sum_k \frac{\partial q_k}{\partial q_i} \frac{\partial p_k}{\partial p_j}$

$$\Rightarrow \{q_i, p_j\}_{q,p} = \sum_k \delta_{ki} \delta_{kj} = \delta_{ij}$$

Where δ_{ij} is a Kronecker delta function

Hence $\{q_i, p_j\} = \delta_{ij}$ $\begin{cases} 1 \; if \; i = j \\ 0 \; if \; i \neq j \end{cases}$ Equal to Kronecker delta function.

6.4 Lagrange's Bracket

$\{u,v\}_{q,p} = \{u,v\}_{Q,P}$ Invariance of Lagrange Bracket

Or $\sum_i \frac{\partial(q_i,p_i)}{\partial(u,v)} = \sum_j \frac{\partial(Q_j,P_j)}{\partial(u,v)}$

Proof: Let q_i and p_i are function of Q_j and P_j

$$\{u,v\}_{q,p} = \sum_i \left(\frac{\partial q_i}{\partial u}\frac{\partial p_i}{\partial v} - \frac{\partial q_i}{\partial v}\frac{\partial p_i}{\partial u}\right) \qquad \text{(I)}$$

And for $p_i = p_i(Q_j, P_j) \quad \Rightarrow \quad \frac{\partial p_i}{\partial v} = \sum_j \left(\frac{\partial p_i}{\partial Q_j}\frac{\partial Q_j}{\partial v} + \frac{\partial p_i}{\partial P_j}\frac{\partial P_j}{\partial v}\right)$

Using Maxwell's equations

$$\frac{\partial p_i}{\partial P_j} = \frac{\partial Q_j}{\partial q_i} \quad \& \quad \frac{\partial p_i}{\partial Q_j} = -\frac{\partial P_j}{\partial q_i}$$ putting in above equation

$$\frac{\partial p_i}{\partial v} = \sum_j \left(-\frac{\partial P_j}{\partial q_i}\frac{\partial Q_j}{\partial v} + \frac{\partial Q_j}{\partial q_i}\frac{\partial P_j}{\partial v}\right) = \sum_j \left(\frac{\partial Q_j}{\partial q_i}\frac{\partial P_j}{\partial v} - \frac{\partial P_j}{\partial q_i}\frac{\partial Q_j}{\partial v}\right) = \{q_i, v\}_{Q,P} \qquad \text{(II)}$$

6.4 Lagrange's Bracket

And $q_i = q_i(Q_j, P_j) \Rightarrow \quad \frac{\partial q_i}{\partial v} = \sum_j \left(\frac{\partial q_i}{\partial Q_j}\frac{\partial Q_j}{\partial v} + \frac{\partial q_i}{\partial P_j}\frac{\partial P_j}{\partial v} \right)$

Using Maxwell's equations

$$\frac{\partial q_i}{\partial Q_j} = \frac{\partial P_j}{\partial p_i} \quad \& \quad \frac{\partial q_i}{\partial P_j} = -\frac{\partial Q_j}{\partial p_i}$$ putting in above equation

$$\frac{\partial q_i}{\partial v} = \sum_j \left(\frac{\partial P_j}{\partial p_i}\frac{\partial Q_j}{\partial v} - \frac{\partial Q_j}{\partial p_i}\frac{\partial P_j}{\partial v} \right) = \{p_i, v\}_{Q,P} \qquad \text{(III)}$$

Putting Eqs (II) &(III) in Eq (I) $\{u, v\}_{q,p} = \sum_i \left(\frac{\partial q_i}{\partial u}\frac{\partial p_i}{\partial v} - \frac{\partial q_i}{\partial v}\frac{\partial p_i}{\partial u} \right)$

$$\{u, v\}_{q,p} = \sum_{ij} \left(\frac{\partial q_i}{\partial u}\left(\frac{\partial Q_j}{\partial q_i}\frac{\partial P_j}{\partial v} - \frac{\partial P_j}{\partial q_i}\frac{\partial Q_j}{\partial v} \right) - \left(\frac{\partial P_j}{\partial p_i}\frac{\partial Q_j}{\partial v} - \frac{\partial Q_j}{\partial p_i}\frac{\partial P_j}{\partial v} \right)\frac{\partial p_i}{\partial u} \right)$$

$$\{u, v\}_{q,p} = \sum_{ij} \left(\frac{\partial P_j}{\partial v}\left(\frac{\partial Q_j}{\partial q_i}\frac{\partial q_i}{\partial u} + \frac{\partial Q_j}{\partial p_i}\frac{\partial p_i}{\partial u} \right) - \frac{\partial Q_j}{\partial v}\left(\frac{\partial P_j}{\partial q_i}\frac{\partial q_i}{\partial u} + \frac{\partial P_j}{\partial p_i}\frac{\partial p_i}{\partial u} \right) \right)$$

6.4 Lagrange's Bracket

$$\{u,v\}_{q,p} = \sum_j \left(\frac{\partial P_j}{\partial v} \sum_i \left(\frac{\partial Q_j}{\partial q_i}\frac{\partial q_i}{\partial u} + \frac{\partial Q_j}{\partial p_i}\frac{\partial p_i}{\partial u} \right) - \frac{\partial Q_j}{\partial v} \sum_i \left(\frac{\partial P_j}{\partial q_i}\frac{\partial q_i}{\partial u} + \frac{\partial P_j}{\partial p_i}\frac{\partial p_i}{\partial u} \right) \right)$$

$$\{u,v\}_{q,p} = \sum_j \left(\frac{\partial P_j}{\partial v}\frac{\partial Q_j}{\partial u} - \frac{\partial Q_j}{\partial v}\frac{\partial P_j}{\partial u} \right)$$

$$\{u,v\}_{q,p} = \sum_j \left(\frac{\partial Q_j}{\partial u}\frac{\partial P_j}{\partial v} - \frac{\partial P_j}{\partial u}\frac{\partial Q_j}{\partial v} \right)$$

$$\{u,v\}_{q,p} = \{u,v\}_{Q,P}$$

6.5 Poisson's Brackets

Poisson's bracket:

- An important binary operation in Hamiltonian mechanics
- Play a central role in Hamilton's equations of motion.
- Distinguishes a class of coordinate transformations (canonical transformations)
- Very useful tool in quantum mechanics and field theory.

The Poisson bracket are defined as

$$[u, v]_{q,p} = \sum_i \left(\frac{\partial u}{\partial q_i} \frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i} \frac{\partial v}{\partial q_i} \right)$$

Or

$$[u, v]_{Q,P} = \sum_i \left(\frac{\partial u}{\partial Q_i} \frac{\partial v}{\partial P_i} - \frac{\partial u}{\partial P_i} \frac{\partial v}{\partial Q_i} \right)$$

Where u, v are two functions, $w.r.t$ the canonical variable (q, p) or (Q, P).

6.5 Poisson's Brackets

Consider f is a function such that

$$f = f(q_i, p_i)$$

$$\frac{df}{dt} = \sum_i \frac{\partial f}{\partial q_i}\frac{dq_i}{dt} + \sum_i \frac{\partial f}{\partial p_i}\frac{dp_i}{dt}$$

$$\frac{df}{dt} = \sum_i \frac{\partial f}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial f}{\partial p_i}\dot{p}_i$$

As we know that $\dot{q}_i = \frac{\partial H}{\partial p}$ and $\dot{p}_i = -\frac{\partial H}{\partial \mathrm{q}}$

Eq 1 can be written as $\frac{df}{dt} = \sum_i \left[\frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i}\right]$

$$\frac{df}{dt} = [f, H]_{q,p}$$

Where $[f, H]$ is called Poisson's Brackets.

6.5 Poisson's Brackets

Generally u and v are two function their Poisson's brackt is

$$[u,v] = \sum_{i=1}^{n}\left[\frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}\right]$$

Properties It is obvious from the definition of the Poisson brackets that

i. $[u,v] = -[v,u]$ Poisson bracket are anti-commutative

ii. $[u,u] = 0 = [v,v]$ Poisson bracket of identical functions are zero.

iii. $[u,c] = 0 = [u,c]$ Where c is independent of p or q.

iv. $[u+v,w] = [u,w] + [v,w]$ Poisson brackets obeys the distributive law

v. $[u,vw] = [u,v]w + v[u,w]$

vi. $[q_i,p_j] = \delta_{ij} \begin{cases} 1 \; if \; i=j \\ 0 \; if \; i \neq j \end{cases}$ Where δ_{if} is the Kronecker delta function.

6.5 Poisson's Brackets

1)Skew symmetric OR anti commutative

(OR show that Poisson brackets do not obey commutative law)

As we know that $$[u,v]_{q,p} = \sum_{i=1}^{n}\left[\frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}\right]$$

Taking –ve sign out of the bracket

$$[u,v]_{q,p} = -\sum_{i=1}^{n}\left[-\frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} + \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}\right]$$

$$[u,v]_{q,p} = -\sum_{i=1}^{n}\left[\frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i} - \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i}\right]$$

$$[u,v]_{q,p} = -\left[\frac{\partial v}{\partial q_i}\frac{\partial u}{\partial p_i} - \frac{\partial v}{\partial p_i}\frac{\partial u}{\partial q_i}\right]$$

$$[u,v]_{q,p} = -[v,u]_{q,p}$$ **proved**

6.5 Poisson's Brackets

For identical function

$$[u,u]_{q,p} = -[v,v]_{q,p} = 0$$

$$[u,u]_{q,p} = \sum_{i=1}^{n}\left[\frac{\partial u}{\partial q_i}\frac{\partial u}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial u}{\partial q_i}\right]$$

Similarly $$[v,v]_{q,p} = 0$$

Poisson brackets with a-constant

Let F be function of generalized q_i and p_i and C is any constant quality.

Then $$[F,C]_{q,p} = \sum_{i=1}^{n}\left[\frac{\partial F}{\partial q_i}\frac{\partial C}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial C}{\partial q_i}\right]$$

Where $$\frac{\partial C}{\partial q_i} = \frac{\partial C}{\partial p_i} = 0$$ where c does not depend on q_i and p_i.

$$[F,C]_{q,p} = 0$$

6.5 Poisson's Brackets

Poisson Brackets obey the distributive law

Consider F_1, F_2 and G are three functions dependently upon q_i andp_i.

Then

$$[(F_1 + F_2), G] = \sum_{i=1}^{n}\left[\frac{\partial(F_1+F_2)}{\partial q_i}\ \frac{\partial G}{\partial p_i} - \frac{\partial(F_1+F_2)}{\partial p_i}\frac{\partial G}{\partial q}\right]$$

$$[(F_1 + F_2), G] = \sum_{i=1}^{n}\left[\left(\frac{\partial F_1}{\partial q_i} + \frac{\partial F_2}{\partial q_i}\right)\ \frac{\partial G}{\partial p_i} - \left(\frac{\partial F_1}{\partial p_i} + \frac{\partial F_2}{\partial p_i}\right).\frac{\partial G}{\partial q_i}\right]$$

$$[(F_1 + F_2), G] = \sum_{i=1}^{n}\left[\left(\frac{\partial F_1}{\partial q_i}\frac{\partial G}{\partial p_i} + \frac{\partial F_2}{\partial q_i}\frac{\partial G}{\partial p_i}\right) - \left(\frac{\partial F_1}{\partial p_i}\ \frac{\partial G}{\partial q_i} + \frac{\partial F_2}{\partial p_i}\frac{\partial G}{\partial q_i}\right)\right]$$

$$[(F_1 + F_2), G] = \sum_{i=1}^{n}\left[\left(\frac{\partial F_1}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F_1}{\partial p_i}\frac{\partial G}{\partial q_i}\right) + \left(\frac{\partial F_2}{\partial q_i}\ \frac{\partial G}{\partial p_i} - \frac{\partial F_2}{\partial p_i}\frac{\partial G}{\partial q_i}\right)\right]$$

$$[(F_1 + F_2), G] = \sum_{i=1}^{n}\left[\left(\frac{\partial F_1}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F_1}{\partial p_i}.\frac{\partial G}{\partial q_i}\right)\right] + \sum_{i=1}^{n}\left[\frac{\partial F_2}{\partial q_i}\ \frac{\partial G}{\partial p_i} - \frac{\partial F_2}{\partial p_i}\frac{\partial G}{\partial q_i}\right]$$

$$[F_1 + F_2, G]_{q,p} = [F_1, G]_{q,p} + [F_2, G]_{q,p}$$

Proved it obeys.

6.5 Poisson's Brackets

Poisson's Brackets are linear

Consider if $F_{(q_i,p_i)}, G_{(q_i,p_i)}$ and $W_{(q_i,p_i)}$ are three functions and a, b are constant, then we here to show that

$$[aF + bG, W] = [aF, W] + [bG, W]$$

$$[aF + bG, W] = a[F, W] + b[G, W]$$

Let $aF = F'$ and $bG = G'$

Then by distributive property we know that

$$[aF + bG, W] = [F' + G', W]_{q,p}$$

$$[F' + G', W] = [F', W] + [G', W]$$

$$[aF + bG, W] = [aF, W] + [bG, W]$$

6.5 Poisson's Brackets

Now consider

$$[aF, W] = \sum_{i=1}^{n} \left[\frac{\partial (aF)}{\partial q_i} \cdot \frac{\partial W}{\partial p_i} - \frac{\partial (aF)}{\partial p_i} \frac{\partial W}{\partial q_i}\right]$$

$$[aF, W] = a \sum_{i=1}^{n} \left[\frac{\partial F}{\partial q_i} \frac{\partial W}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial W}{\partial q_i}\right]$$

$$[aF, W] = a\,[F, W]$$

Similarly $$[bG, W] = b\,[G, W]$$

Hence

$$[aF + bG, W] = a[F, W] + b[G, W]$$

6.5 Poisson's Brackets

Show that if u, v and w are functions dependent on (q_i, p_i), then show that

$$[u, vw] = [u, v]w + v[u, w]$$

As $$[u, vw]_{q,p} = \sum_{i=1}^{n}\left[\frac{\partial u}{\partial q_i}\frac{\partial (vw)}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial (vw)}{\partial q_i}\right]$$

$$[u, vw]_{q,p} = \sum_{i=1}^{n}\left[\frac{\partial u}{\partial q_i}\left(v\frac{\partial w}{\partial p_i} + w\frac{\partial v}{\partial p_i}\right) - \frac{\partial u}{\partial p_i}\left(v\frac{\partial w}{\partial q_i} + w\frac{\partial v}{\partial q_i}\right)\right]$$

$$[u, vw]_{q,p} = \sum_{i=1}^{n}\left[\left(w\frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - w\frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}\right) + \left(v\frac{\partial u}{\partial q_i}\frac{\partial w}{\partial p_i} - v\frac{\partial u}{\partial p_i}\frac{\partial w}{\partial q_i}\right)\right]$$

$$[u, vw]_{q,p} = \sum_{i=1}^{n} w\left[\frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}\right] + v\left[\frac{\partial u}{\partial q_i}\frac{\partial w}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial w}{\partial q_i}\right]$$

$$[u, vw]_{p,q} = w[u, v]_{q,p} + v[u, w]_{q,p}$$

6.5 Poisson's Brackets

Proved that $$\frac{\partial}{\partial t}[F,G] = \left[\frac{\partial F}{\partial t}, G\right] + \left[F, \frac{dG}{dt}\right]$$

Sol: $$[F,G] = \sum_{i=1}^{n}\left[\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right]$$

$$\frac{\partial}{\partial t}[F,G] = \frac{\partial}{\partial t}\sum_{i=1}^{n}\left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right)$$

$$\frac{\partial}{\partial t}[F,G] = \sum_{i=1}^{n}\left[\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i}\right) - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right)\right]$$

$$\frac{\partial}{\partial t}[F,G] = \sum_{i=1}^{n}\left[\frac{\partial F}{\partial q_i}\frac{\partial}{\partial t}\left(\frac{\partial G}{\partial p_i}\right) + \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial q_i}\right)\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial}{\partial t}\left(\frac{\partial G}{\partial q_i}\right) - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial p_i}\right)\frac{\partial G}{\partial q_i}\right]$$

$$\frac{\partial}{\partial t}[F,G] = \sum_{i=1}^{n}\left[\frac{\partial F}{\partial q_i}\frac{\partial}{\partial t}\left(\frac{\partial G}{\partial p_i}\right) - \frac{\partial F}{\partial p_i}\frac{\partial}{\partial t}\left(\frac{\partial G}{\partial q_i}\right)\right] + \sum_{i=1}^{n}\left[\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial q_i}\right)\frac{\partial G}{\partial p_i} - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial p_i}\right)\frac{\partial G}{\partial q_i}\right]$$

6.5 Poisson's Brackets

$$\frac{\partial}{\partial t}[F,G] == \sum_{i=1}^{n}\left[\frac{\partial F}{\partial q_i}\frac{\partial}{\partial p_i}\left(\frac{\partial G}{\partial t}\right) - \frac{\partial F}{\partial p_i}\frac{\partial}{\partial q_i}\left(\frac{\partial G}{\partial t}\right)\right] + \sum_{i=1}^{n}\left[\frac{\partial}{\partial q_i}\left(\frac{\partial F}{\partial t}\right)\frac{\partial G}{\partial p_i} - \frac{\partial}{\partial p_i}\left(\frac{\partial F}{\partial t}\right)\frac{\partial G}{\partial q_i}\right]$$

$$\frac{\partial}{\partial t}[F,G] = \left[F,\frac{\partial G}{\partial t}\right] + \left[\frac{\partial F}{\partial t},G\right]$$ ***proved***

Assignment **Show that**

i. $[q_i, q_j] = [p_i, p_j] = 0$

ii. $[q_i, p_j] = \delta_{ij}\begin{cases}1 & if \quad i = j\\ 0 & if \quad i \neq j\end{cases}$

$$where\ \frac{\partial p}{\partial q} = 0 \qquad \frac{\partial q}{\partial p} = 0$$

6.5 Poisson's Brackets

Show that Poisson's brackets is invariant under canonical transformation

$$[F,G]_{q,p} = [F,G]_{Q,P}$$

Proof: let F and G be two arbitrary function of q and p.Then

$$[F,G]_{q,p} = \sum_i \left[\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right]$$

Let the Transformation $Q = Q(q,p)$ and $P = P(q,p)$

Inverse transformation $q = q(Q,P)$ and $p = p(Q,P)$

Therefore $G = G(Q,P)$

$$\partial G = \sum_k \frac{\partial G}{\partial Q_k}\partial Q_K + \sum_k \frac{\partial G}{\partial P_k}\partial P_k$$

Then $\frac{\partial G}{\partial q_i} = \sum_k \frac{\partial G}{\partial Q_k}\frac{\partial Q_k}{\partial q_i} + \frac{\partial G}{\partial P_k}\frac{\partial P_k}{\partial q_i}$ & $\frac{\partial G}{\partial p_i} = \sum_k \frac{\partial G}{\partial Q_k}\frac{\partial Q_K}{\partial p_i} + \sum_k \frac{\partial G}{\partial P_K}\frac{\partial P_k}{\partial P_i}$

6.5 Poisson's Brackets

$$\Rightarrow [F,G]_{q,p} = \sum_{i,k}\left[\frac{\partial F}{\partial q_i}\left(\frac{\partial G}{\partial Q_k}\frac{\partial Q_k}{\partial p_i}+\frac{\partial G}{\partial P_k}\frac{\partial P_k}{\partial p_i}\right)-\frac{\partial F}{\partial p_i}\left(\frac{\partial G}{\partial Q_k}\frac{\partial Q_k}{\partial q_i}+\frac{\partial G}{\partial P_k}\frac{\partial P_k}{\partial q_i}\right)\right]$$

$$\Rightarrow [F,G]_{q,p} = \sum_{i,k}\left[\frac{\partial F}{\partial q_i}\left(\frac{\partial G}{\partial Q_k}\frac{\partial Q_k}{\partial p_i}\right)-\frac{\partial F}{\partial p_i}\left(\frac{\partial G}{\partial Q_k}\frac{\partial Q_k}{\partial q_i}\right)+\frac{\partial F}{\partial q_i}\left(\frac{\partial G}{\partial P_k}\frac{\partial P_k}{\partial p_i}\right)-\frac{\partial F}{\partial p_i}\left(\frac{\partial G}{\partial P_k}\frac{\partial P_k}{\partial q_i}\right)\right]$$

$$\Rightarrow [F,G]_{q,p} = \sum_{k}\left[\frac{\partial G}{\partial Q_k}\left(\sum_i\left[\frac{\partial F}{\partial q_i}\frac{\partial Q_k}{\partial p_i}-\frac{\partial F}{\partial p_i}\frac{\partial Q_k}{\partial q_i}\right]\right)+\frac{\partial G}{\partial P_k}\left(\sum_i\left[\frac{\partial F}{\partial q_i}\frac{\partial P_k}{\partial p_i}-\frac{\partial F}{\partial p_i}\frac{\partial P_k}{\partial q_i}\right]\right)\right]$$

$$\Rightarrow [F,G]_{q,p} = \sum_{k}\left[\frac{\partial G}{\partial Q_k}[F,Q_k]_{q,p}+\frac{\partial G}{\partial P_k}[F,P_k]_{q,p}\right] \qquad \text{(I)}$$

Replacing F by Q_i

0

$$\Rightarrow [Q_i,G]_{q,p} = \sum_{k}\left[\frac{\partial G}{\partial Q_k}[Q_i,Q_k]_{q,p}+\frac{\partial G}{\partial P_k}[Q_i,P_k]_{q,p}\right]$$

$$\Rightarrow [Q_i,G]_{q,p} = \sum_{k}\frac{\partial G}{\partial P_k}\delta_{ik} = \frac{\partial G}{\partial P_i}$$

6.5 Poisson's Brackets

Now replacing G by F in previous equation

$$\Rightarrow [Q_i, F] = \frac{\partial F}{\partial P_i}$$

Or $$\Rightarrow [F, Q_k]_{q,p} = -\frac{\partial F}{\partial P_k} \qquad \text{(II)}$$

Similarly, replacing F by P_i in eq (I)

$$\Rightarrow [P_i, G]_{q,p} = \sum_k \left[\frac{\partial G}{\partial Q_k}[P_i, Q_k]_{q,p} + \frac{\partial G}{\partial P_k}[P_i, P_k]_{q,p}\right]$$

0

$$\Rightarrow [P_i, G]_{q,p} = \sum_k \frac{\partial G}{\partial Q_k}[P_i, Q_k]$$

$$\Rightarrow [P_i, G]_{q,p} = -\sum_k \frac{\partial G}{\partial Q_k}[Q_k, P_i]$$

6.5 Poisson's Brackets

$$\Rightarrow [P_i, G]_{q,p} = -\sum_k \frac{\partial G}{\partial Q_k}\delta_{ik} = -\frac{\partial G}{\partial Q_k} \qquad \text{for} \qquad i = k$$

And $\Rightarrow [G, P_k]_{q,p} = \frac{\partial G}{\partial Q_k}$

Now replacing G by F, we get $[F, P_k] = \frac{\partial F}{\partial Q_k}$ (III)

Putting a and b in eq 1

$$\Rightarrow [F, G]_{q,p} = \sum_k \left[\frac{\partial G}{\partial Q_k}[F, Q_k]_{q,p} + \frac{\partial G}{\partial P_k}[F, P_k]_{q,p}\right] = \sum_k \left(-\frac{\partial G}{\partial Q_k}\frac{\partial F}{\partial P_k} + \frac{\partial G}{\partial P_k}\frac{\partial F}{\partial Q_k}\right)$$

$$\Rightarrow [F, G]_{q,p} = \sum_k \left(\frac{\partial F}{\partial Q_k}\frac{\partial G}{\partial P_k} - \frac{\partial F}{\partial P_k}\frac{\partial G}{\partial Q_k}\right) = [F, G]_{P,Q}$$

$$\Rightarrow [F, G]_{q,p} = [F, G]_{P,Q}$$

6.5 Jacobi Identity

$$[u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0$$

Solution Since $[u,v] = \sum_{k=1}^{n}\left(\frac{\partial u}{\partial q_k}\frac{\partial v}{\partial p_k} - \frac{\partial u}{\partial p_k}\frac{\partial v}{\partial q_k}\right)$

$$[u,v] = \sum_{k=1}^{n}\left(\frac{\partial u}{\partial q_k}\frac{\partial}{\partial p_k} - \frac{\partial u}{\partial p_k}\frac{\partial}{\partial q_k}\right)v$$

$$[u,v] = D_u v$$

Where $D_u = \sum_i^{2n} \alpha_i \frac{\partial}{\partial \xi_i}$

Similarly $D_v = \sum_j^{2n} \beta_j \frac{\partial}{\partial \xi_j}$

Note

For 1 to n $\alpha_i = \frac{\partial u}{\partial q_k}$ & $\frac{\partial}{\partial \xi_i} = \frac{\partial}{\partial p_k}$

For n+1 to 2n $\alpha_i = \frac{\partial u}{\partial p_k}$ & $\frac{\partial}{\partial \xi_i} = \frac{\partial}{\partial q_k}$

Now considering first two terms

$$[u,[v,w]] + [v,[w,u]] = [u,[v,w]] - [v,[u,w]] = [u, D_v w] - [v, D_u w]$$

6.5 Jacobi Identity

$$\Rightarrow [u,[v,w]] + [v,[w,u]] = [u, D_v w] - [v, D_u w] = D_u(D_v w) - D_v(D_u w)$$

$$\Rightarrow [u,[v,w]] + [v,[w,u]] = \sum_i^{2n} \alpha_i \frac{\partial}{\partial \xi_i}\left(\sum_j^{2n} \beta_j \frac{\partial}{\partial \xi_j} w\right) - \sum_j^{2n} \beta_j \frac{\partial}{\partial \xi_j}\left(\sum_i^{2n} \alpha_i \frac{\partial}{\partial \xi_i} w\right)$$

$$\Rightarrow [u,[v,w]] + [v,[w,u]] = \sum_{i,j}^{2n} \alpha_i \beta_j \frac{\partial^2 w}{\partial \xi_i \partial \xi_j} + \sum_{i,j}^{2n} \alpha_i \frac{\partial \beta_j}{\partial \xi_i}\frac{\partial w}{\partial \xi_j} - \sum_{i,j}^{2n} \alpha_i \beta_j \frac{\partial^2 w}{\partial \xi_j \partial \xi_i} - \sum_{i,j}^{2n} \beta_j \frac{\partial \alpha_i}{\partial \xi_j}\frac{\partial w}{\partial \xi_i}$$

$$\Rightarrow [u,[v,w]] + [v,[w,u]] = \sum_{i,j}^{2n} \alpha_i \frac{\partial \beta_j}{\partial \xi_i}\frac{\partial w}{\partial \xi_j} - \sum_{i,j}^{2n} \beta_j \frac{\partial \alpha_i}{\partial \xi_j}\frac{\partial w}{\partial \xi_i}$$

$$\Rightarrow [u,[v,w]] + [v,[w,u]] = \sum_{i,j}^{2n} \left(\alpha_i \frac{\partial \beta_j}{\partial \xi_i} - \beta_i \frac{\partial \alpha_j}{\partial \xi_i}\right)\frac{\partial w}{\partial \xi_j}$$

By using the property that sum is not effected if the indices are interchanged (dummy indices)

6.5 Jacobi Identity

$$\Rightarrow \sum_{i,j}^{2n}\left(\alpha_i \frac{\partial \beta_j}{\partial \xi_i} - \beta_i \frac{\partial \alpha_j}{\partial \xi_i}\right)\frac{\partial w}{\partial \xi_j} = \sum_{i,j}^{n}\left(\alpha_i \frac{\partial \beta_j}{\partial \xi_i} - \beta_i \frac{\partial \alpha_j}{\partial \xi_i}\right)\frac{\partial w}{\partial \xi_j} + \sum_{i,j=n+1}^{2n}\left(\alpha_i \frac{\partial \beta_j}{\partial \xi_i} - \beta_i \frac{\partial \alpha_j}{\partial \xi_i}\right)\frac{\partial w}{\partial \xi_j}$$

$$\Rightarrow \sum_{i,j}^{2n}\left(\alpha_i \frac{\partial \beta_j}{\partial \xi_i} - \beta_i \frac{\partial \alpha_j}{\partial \xi_i}\right)\frac{\partial w}{\partial \xi_j} = \sum_i \left(A_j \frac{\partial w}{\partial p_i} + B_j \frac{\partial w}{\partial q_i}\right)$$

Replacing w by $p_j \Rightarrow \quad \sum_i \left(A_j \frac{\partial p_j}{\partial p_i} + B_j \frac{\partial p_j}{\partial q_i}\right) = \sum_i A_j \frac{\partial p_j}{\partial p_i} = \sum_i A_i \delta_{ij} = A_j$

$$\Rightarrow \left[u,\left[v,p_j\right]\right] - \left[v,\left[u,p_j\right]\right] = A_j$$

$$\Rightarrow A_j = \left[u,\frac{\partial v}{\partial q_j}\right] - \left[v,\frac{\partial u}{\partial q_j}\right]$$

$$\Rightarrow A_j = \left[u,\frac{\partial v}{\partial q_j}\right] + \left[\frac{\partial u}{\partial q_j},v\right] = \frac{\partial}{\partial q_j}[u,v]$$

6.5 Jacobi Identity

Replacing w by q_j

$$\left[u,\left[v,q_j\right]\right]-\left[v,\left[u,q_j\right]\right]=B_j$$

$$\Rightarrow B_j=-\left[u,\frac{\partial v}{\partial p_j}\right]+\left[v,\frac{\partial u}{\partial p_j}\right]$$

$$\Rightarrow B_j=-\frac{\partial}{\partial p_j}[u,v]$$

$$\Rightarrow [u,[v,w]]+[v,[w,u]]=\sum_i\left[\frac{\partial w}{\partial p_i}\frac{\partial}{\partial q_j}[u,v]-\frac{\partial w}{\partial q_j}\frac{\partial}{\partial p_j}[u,v]\right]$$

$$\Rightarrow [u,[v,w]]+[v,[w,u]]=-[w,[u,v]]$$

$$\Rightarrow [u,[v,w]]+[v,[w,u]]+[w,[u,v]]=0$$

6.5 Poisson’s Brackets

Evaluate the following Poisson bracket and explain its physical meaning:

$$\left[mr^2\dot{\vartheta}^2,\ \frac{m}{2}\left(p_r^2+\frac{p_\vartheta^2}{r^2}+\frac{p_\varphi^2}{r^2\sin^2\vartheta}\right)+V(r)\right]_{r,\vartheta}$$

Find the value of $\alpha \in \mathbb{R}$ for which the following transformation is canonical:

$$Q(p,q)=\ln\left(\frac{p}{2q}\right);\ P(p,q)=-\frac{1}{2}qp^\alpha$$

$$
\begin{aligned}
\vec{L} &= \vec{r} \times \vec{p} \\
L_x &= y p_z - z p_y \\
L_y &= z p_x - x p_z \\
L_z &= x p_y - y p_x
\end{aligned}
$$

$$\text{Poisson} \quad \Rightarrow [L_x, L_y] = \text{some conserved quantity}$$

We will need some partial derivatives to compute the PB of L_x and L_y.

$$
\begin{aligned}
\frac{\partial L_x}{\partial \vec{r}} &= \{0, p_z, -p_y\} \quad , \quad \frac{\partial L_x}{\partial \vec{p}} = \{0, -z, y\} \\
\frac{\partial L_y}{\partial \vec{r}} &= \{-p_z, 0, p_x\} \quad , \quad \frac{\partial L_y}{\partial \vec{p}} = \{z, 0, -x\}
\end{aligned}
$$

$$[L_x, L_y] = \frac{\partial L_x}{\partial \vec{p}} \cdot \frac{\partial L_y}{\partial \vec{r}} - \frac{\partial L_x}{\partial \vec{r}} \cdot \frac{\partial L_y}{\partial \vec{p}} = y p_x - x p_y = -L_z$$

$$\text{if} \quad [H, L_z] = 0 \Rightarrow L_z = \text{const}$$
$$[T + U, L_z] = [T, L_z] + [U, L_z]$$

Let's do these one at a time...

$$\begin{aligned}[T, L_z] &= [T, xp_y - yp_x] = [T, xp_y] - [T, yp_x] \\ &= ([T, x]p_y + [T, p_y]x) - ([T, y]p_x + [T, p_x]y)\end{aligned}$$

It doesn't look like we are winning, but what I am doing is breaking this down into small enough parts that I can use my identities. For instance,

$$\begin{aligned}[T, p_x] &= \frac{1}{2m}[p_x^2 + p_y^2 + p_z^2, p_x] \\ [p_x^2, p_x] &= 2p_x[p_x, p_x] = 0 \\ \Rightarrow &\quad [T, p_i] = 0 \quad \forall i\end{aligned}$$

$$[T, x] = \frac{1}{2m}[p_x^2, x] = \frac{p_x}{m}[p_x, x] = \frac{p_x}{m}$$

note that $[p_y, x] = [p_z, x] = 0$. Finally,

$$[T, L_z] = \frac{p_x p_y}{m} - \frac{p_y p_x}{m} = 0$$

$\Rightarrow L_z$ conserved for free particle! (and L_x and L_y)

Well, I guess we knew that. Let's do U...

$$\begin{aligned} [U, L_z] &= ([U, x]p_y + [U, p_y]x) - ([U, y]p_x + [U, p_x]y) \\ &= [U, p_y]x - [U, p_x]y \end{aligned}$$

where I have dropped 2 terms since $[q_j, q_k] = 0$ and $U(r)$ has no p_i in it.

$$\begin{aligned} [U, p_y] &= -\frac{\partial U}{\partial y} = -\frac{\partial U}{\partial r}\frac{\partial r}{\partial y} \\ &= -\frac{\partial U}{\partial r}\frac{y}{r} \end{aligned}$$

$$\text{where} \qquad r = \sqrt{x^2 + y^2 + z^2} \Rightarrow \frac{\partial r}{\partial y} = \frac{1}{2r}2y = \frac{y}{r}$$

Putting these together to find the PB of L_z with U,

$$\begin{aligned} [U, L_z] &= [U, p_y]x - [U, p_x]y \\ &= -\frac{\partial U}{\partial r}\left(\frac{y}{r}x - \frac{x}{r}y\right) = 0 \end{aligned}$$